On Paraconsistent Belief Revision

Rafael R. Testa

Centre for Logic, Epistemology and History of Science
State University of Campinas
Brazil

joint work with Marcelo Coniglio and Márcio Ribeiro

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Paraconsistency

In classical logic, contradictoriness (the presence of contradictions in a theory) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable. This is an effect of a logical property known as *explosiveness* (*ex falso quodlibet* or *ex contradictione sequitur quodlibet*, that is, anything follows from a contradiction). Paraconsistent logics are precisely the logics that challenge this assumption by rejecting the classical consistency presupposition.

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LFIs

The Logics of Formal Inconsistency (**LFI**s) [Carnielli, Coniglio & Marcos 2007] constitute the class of paraconsistent logics which can internalize the meta-theoretical notions of consistency and inconsistency. As a consequence, despite constituting fragments of consistent logics, the **LFI**s can canonically be used to faithfully encode all consistent inferences.

Roughly, the idea in the **LFI**s is to express the meta-theoretical notions of consistency and inconsistency at the object language level, by adding to the language a new connective.

(1) **Explosion Principle** $\alpha, \neg\alpha \vdash \beta$ is not the case in general

(2) **Gentle Explosion Principle** $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ is always the case.

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Systems

Two systems of *Paraconsistent Belief Revision* are defined: AGMp and AGM$\circ$ [Testa 2014].

Both systems are defined over Logics of Formal Inconsistency, but the constructions of the second are specially related to the formal consistency operator.

The mbC

Definition (**mbC[?]**)

Axioms:

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \delta)) \to (\alpha \to \delta))$

(A3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(A4) $(\alpha \wedge \beta) \to \alpha$

(A5) $(\alpha \wedge \beta) \to \beta$

(A6) $\alpha \to (\alpha \vee \beta)$

(A7) $\beta \to (\alpha \vee \beta)$

(A8) $(\alpha \to \delta) \to ((\beta \to \delta) \to ((\alpha \vee \beta) \to \delta))$

(A9) $\alpha \vee (\alpha \to \beta)$

(A10) $\alpha \vee \neg\alpha$

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

Inference Rule:

(Modus Ponens) $\alpha, \alpha \to \beta \vdash \beta$

Why a paraconsistent system?

Classical AGM adopts the following rationality criteria [Gärdenfors and Rott, 1995]:
(non-contradictoriness) Where possible, epistemic states should remain non-contradictory;
(Cclosure) Any sentence logically entailed by beliefs in an epistemic state should be included in the epistemic state;
(minimality) When changing epistemic states, loss of information should be kept to a minimum;

Revisions

Definition (Internal Revision)

$K * \alpha = (K - \neg\alpha) + \alpha$

Definition (External Revision (Hansson 1993))

$K * \alpha = (K + \alpha) - \neg\alpha$

A new system from the sketch?

AGM compliance

An AGM-compliant logic is simply one in which is possible to completely characterize the contraction operation via the classical postulates. Formally we have the following:

Definition (AGM-compliance (Flouris 2006))

A logic $\mathbf{L}$ is AGM-compliant if it admits at least one operation $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ on $\mathbf{L}$ which satisfies the postulates for contraction.

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LFIs are AGM-compliant

Compact and supra-classical logics such as the **LFI**s considered here are AGM-compliant.
Furthermore, in this kind of logic *recovery* ($K \subseteq (K - \alpha) + \alpha$) and *relevance* (if $\beta \in K \setminus K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$ and $\alpha \in K' + \beta$) are equivalent.
Hence, altough this is not valid in general, relevance and recovery can be used indistinguishably for the logics considered here [Ribeiro, Wassermann and Flouris 2013].

AGMp system

Definition (AGMp external revision)

An AGMp external revision over **L** is an operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:

(closure) $K * \alpha = Cn(K * \alpha)$

(success) $\alpha \in K * \alpha$

(inclusion) $K * \alpha \subseteq K + \alpha$

(vacuity) if $\neg\alpha \notin K$ then $K + \alpha \subseteq K * \alpha$

(non-contradiction) if $\neg\alpha \in K * \alpha$ then $\vdash \neg\alpha$

(relevance) if $\beta \in K \setminus (K * \alpha)$ then there exists X such that

$$K * \alpha \subseteq X \subseteq K + \alpha,\ \neg\alpha \notin Cn(X) \text{ and } \neg\alpha \in Cn(X) + \beta$$

(pre-expansion) $(K + \alpha) * \alpha = K * \alpha$

Representation Theorem

Given the definition of partial meet contraction, as expected external partial meet revision is fully characterized by the postulates of Definition 5.

Theorem

An operation $* : Th(\mathbf{L}) \times \mathbb{L} \rightarrow Th(\mathbf{L})$ *is an AGMp external revision over* $\mathbf{L}$ *iff it is an external partial meet revision operator over* $\mathbf{L}$*, that is: there is a selection function* γ *for AGMp in* $\mathbf{L}$ *such that* $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$*, for every* K *and* α*.*

AGM$\circ$ system

Definition (Postulates for AGM$\circ$ contraction)

A contraction over **L** is a function $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:

(closure) $K - \alpha = Cn(K - \alpha)$.

(success) If $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$ then $\alpha \notin K - \alpha$.

(inclusion) $K - \alpha \subseteq K$.

(failure) If $\circ\alpha \in K$ then $K - \alpha = K$.

(relevance) If $\beta \in K \setminus K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$ and $\alpha \in K' + \beta$.

Definition (selection function for AGM$\circ$ contraction)

A selection function in **L** is a function
γ : $Th(\mathbf{L}) \times \mathbf{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ such that, for every K and α:

1. $\gamma(K, \alpha) \subseteq K \perp \alpha$ if $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$.
2. $\gamma(K, \alpha) = \{K\}$ otherwise.

The partial meet contraction is the intersection of the sets selected by the choice function:

$$K -_\gamma \alpha = \bigcap \gamma(K, \alpha).$$

Theorem (Representation for AGM$\circ$ contraction)

An operation $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfies the postulates of Definition 7 iff there exists a selection function* γ *in* $\mathbf{L}$ *such that* $K - \alpha = \bigcap \gamma(K, \alpha)$*, for every* K *and* α*.*

Definition (**Postulates for internal AGM$\circ$ revision**)

An internal AGM$\circ$ revision over **L** is an operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following:

(closure) $K * \alpha = Cn(K * \alpha)$.

(success) $\alpha \in K * \alpha$.

(inclusion) $K * \alpha \subseteq K + \alpha$.

(non-contradiction) If $\neg\alpha \notin Cn(\emptyset)$ and $\circ\neg\alpha \notin K$ then $\neg\alpha \notin K * \alpha$.

(failure) If $\circ\neg\alpha \in K$ then $K * \alpha = K + \alpha$

(relevance) If $\beta \in K \setminus K * \alpha$ then there exists K' such that $K \cap K * \alpha \subseteq K' \subseteq K$ and $\neg\alpha \notin K'$, but $\neg\alpha \in K' + \beta$.

Theorem (Representation for internal AGM$\circ$ partial meet revision)

An operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates of Definition 10 if and only if there exists a selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha$, *for every* K *and* α.

Definition (**Postulates for external AGM$\circ$ revision**)

An external revision over **L** is a function
$* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:

(closure) $K * \alpha = Cn(K * \alpha)$.

(success) $\alpha \in K * \alpha$.

(inclusion) $K * \alpha \subseteq K + \alpha$.

(non-contradiction) if $\neg\alpha \notin Cn(\emptyset)$ and $\sim\alpha \notin K$ then $\neg\alpha \notin K * \alpha$.

(failure) If $\sim\alpha \in K$ then $K * \alpha = \mathbb{L}$

(relevance) If $\beta \in K \setminus K * \alpha$ then there exists K' such that $K * \alpha \subseteq K' \subseteq K + \alpha$ and $\neg\alpha \notin K'$, but $\neg\alpha \in K' + \beta$.

(pre-expansion) $(K + \alpha) * \alpha = K * \alpha$.

Theorem (**Representation for external AGM$\circ$ partial meet revision**)

An operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates for external partial meet AGM*$\circ$ *revision (see Definition 12) iff there is a selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$*, for every* K *and* α*.*

The logical possibility of defining an external revision operator over **L** challenges the need of a prior contraction, as in the internal revision. Thus, it is possible to interpret the contraction underlying an internal revision as an unnecessary retraction and therefore as a violation of the principle of minimality. On the other hand, if we consider the non-contradiction principle as a priority, then the internal revision remains to be the only rational option. This illustrates the clear opposition between the principle of non-contradiction and that of minimality. Such opposition deserves further attention in future works.
By capturing two different principles of rationality, both revisions differ both intuitively and logically.

Consolidation and semi-revision

Definition (**Remainder for sets**)

Let K be a belief set in **L** and $A \subset \mathbb{L}$. The set $K \perp_P A \subseteq \wp(\mathbb{L})$ is such that for all $X \subseteq \mathbb{L}$, $X \in K \perp_P A$ iff the following is the case:

1. $X \subseteq K$
2. $A \cap Cn(X) = \emptyset$
3. If $X \subset X' \subseteq K$ then $A \cap Cn(X') \neq \emptyset$.

Consolidation considers a specific subset A, that is, the one that represents the totality of contradictory sentences in K, defined as follows:

Definition (**Contradictory set**)

Let K be a belief set in **L**. The set Ω_K of contradictory sentences of K. is defined as follows:

$$\Omega_K = \{\alpha \in K : \text{ exists } \beta \in \mathbb{L} \text{ such that } \alpha = \beta \wedge \neg\beta\}.$$

Definition (**Consolidation function**)

A *consolidation function* in **L** is a function $\gamma : Th(\mathbf{L}) \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ such that, for every belief set K in **L**:

1. If $K \neq \mathbb{L}$ then $\gamma(K) \subseteq K \bot_P \Omega_K$
2. If $K = \mathbb{L}$ then $\gamma(K) = \{K\}$

The consolidation operator defined by a consolidation function γ is then defined as follows: for every belief set K in **L**,

$$K!_\gamma = \bigcap \gamma(K)$$

As stated previously, both revisions require effective integration of the new belief. On the other hand, from the definition of external revision, it is possible to define a revision in which the *principle of primacy of new information*, tacitly accepted in internal and external revisions, is challenged. In the context of belief bases it is called *semi-revision* by Hansson, which is characterized by the expansion-consolidation scheme.

The semi-revision for belief sets can be defined as a generalization of external-revision, in which the choice for the removal is left to the selection function.

$$K?_{\gamma}\alpha = (K + \alpha)!_{\gamma}$$

Final remarks...